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GRAPHS OF GIVEN GENUS AND ARBITRARILY LARGE MAXIMUM GENUS

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Abstract. The *maximum genus* of a connected graph G is the maximum among the genera of all compact orientable 2-manifolds upon which G has 2-cell embeddings. In the theorems that follow the use of an edge-adding technique is combined with the well-known Edmonds' technique to produce the desired results. Planar graphs of arbitrarily large maximum genus are displayed in Theorem 1. Theorem 2 shows that the possibility for arbitrarily large difference between genus and maximum genus is not limited to planar graphs. In particular, we show that the wheel graph, the standard maximal planar graph, and the prism graph are upper embeddable. We then show that given m and n, there is a graph of genus n and maximum genus larger than mn.

The *maximum genus* $\gamma_M(G)$ of a connected graph G is the maximum among the genera of all compact orientable 2-manifolds upon which G has 2-cell embeddings. In [4], the authors characterized those graphs which have maximum genus equal to ordinary genus. Another question concerning the relationship between γ and γ_M is the following: Does the genus of a graph somehow bound its maximum genus? In this paper we first display certain planar graphs of arbitrarily large maximum genus and then show that the possibility for arbitrarily large difference between genus and maximum genus is not limited to planar graphs.

When an edge e or a vertex v of an embedded graph is on the boundary of a face F, we say that e (or v) is in the face F. The Betti number, $\beta(G)$, of a connected graph G is given by $\beta(G) = E - V + 1$, where E and V are the number of edges and vertices of G, respectively. For a real number x, $[x]$ is the greatest integer smaller than or equal to x. For two vertices a and b, (a, b) will denote the directed edge from a to b,

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while $[a, b]$ will denote the non-directed edge. Nordhaus, Stewart and White [5] give the following upper bound theorem.

Theorem. *An upper bound for the maximum genus of an arbitrary connected graph G is given by* $\gamma_M(G) \leq [\frac{1}{2}\beta(G)]$. *Equality holds if and only if the embedding has one or two faces, according as* $\beta(G)$ *is even or odd, respectively.*

Definition. A graph is *upper embeddable* if $\gamma_M(G) = [\frac{1}{2}\beta(G)]$.

Many of the proofs which follow employ the following edge-adding technique [6]:

Theorem. *Let G be a connected graph and let i and j denote non-adjacent vertices of G. Let T be a 2-cell embedding of G which has vertex i in face* F_i *and vertex j in face* F_j. *Let G′ be the graph G with the edge* $[i, j]$ *added. Then:*

(i) *If* $F_i \neq F_j$, *then G′ has a 2-cell embedding with one less face than T.*

(ii) *If* $F_i = F_j$, *then G′ has a 2-cell embedding T′ with one more face than T. Furthermore, the directed edges* (i, j) *and* (j, i) *appear in different faces of T′.*

Let C_m be a cycle with m vertices, P_m a path with m vertices and K_m the complete graph on m vertices. Then the wheel graph W_n, the standard maximal planar graph M_n, and the prism graph R_{2n} are given by the equations (where + is the join and × is the Cartesian product as defined in [3]):

$$W_n = K_1 + C_{n-1}, \quad n \geq 4,$$

$$M_n = K_2 + P_{n-2}, \quad n \geq 4,$$

$$R_{2n} = C_n \times K_2, \quad n \geq 3.$$

The Betti numbers of these graphs are $\beta(W_n) = n - 1$, $\beta(M_n) = 2n - 5$, and $\beta(R_{2n}) = n + 1$.

Theorem 1. *W_n, M_n and R_{2n} are upper embeddable graphs for each n where defined.*

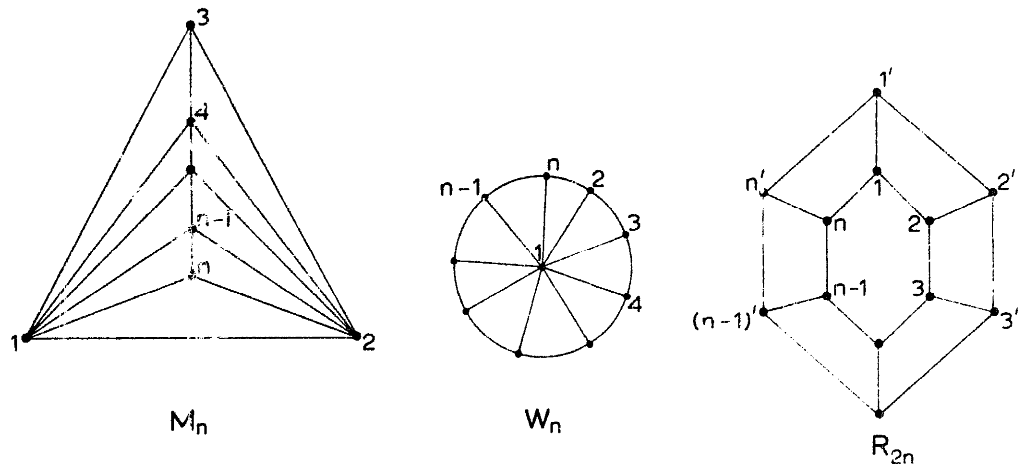

Fig. 1.

Proof. We label the graphs as shown in Fig. 1. It is useful to first recall that K_4 has a 2-cell embedding with two faces on the torus, and for which each vertex is in both of the faces [5].

Case 1: the graph M_n, $n \geq 4$. M_4 is K_4 and is thus upper embeddable. For all $n \geq 5$, if one uses the following vertex permutations in an Edmonds' scheme ([2], see also [7]), he obtains an embedding with two faces for M_n:

$$P_1: (2\ 3\ 4\ \dots\ n)\,,$$

$$P_2: (1\ 3\ 4\ \dots\ n)\,,$$

$$P_3: (4\ 1\ 2)\,,$$

$$P_n: ((n-1)\ 1\ 2)\,,$$

$$P_j: ((j-1)(j+1)\ 1\ 2); \quad j = 4, \dots, n-1.$$

Case 2: the graph W_n, $n \geq 4$. W_4 is again K_4. Thus the center of W_4 is in both faces of its two-face embedding. We assume the induction hypothesis that W_n is upper embeddable and that, if n is even, the center "1" of W_n is in both faces of the two-face embedding.

In the maximal embedding for W_n, subdivide the edge $[n, 2]$ with the vertex $n+1$. Let F be a face which contains this new vertex. If $\beta(W_n)$ is odd, there is a face F' containing "1" so that $F' \neq F$. Hence the edge $[1, n+1]$ may be added so that the resultant graph W_{n+1} has an embedding with one face and is thus upper embeddable. If $\beta(W_n)$ is even, the addition of edge $[1, n+1]$ yields a two-face embedding for W_{n+1} which has the vertex "1" in both faces. Again W_{n+1} is upper embeddable. By induction, the wheel graph is upper embeddable.

Case 3: the graph R_{2n}, $n \geq 3$. The following vertex permutations give an embedding of R_6 with one face:

$$P_1 : (2\ 1'\ 3)\,,$$

$$P_2 : (1\ 2'\ 3)\,,$$

$$P_3 : (2\ 3'\ 1)\,,$$

$$P_{1'} : (2'\ 1\ 3')\,,$$

$$P_{2'} : (1'\ 2\ 3')\,,$$

$$P_{3'} : (3\ 2'\ 1')\,.$$

We proceed inductively, assuming that R_{2k} is upper embeddable.

If $k+1$ is even, R_{2k} has a 2-cell embedding with one face. For this maximal embedding, use the new vertices $k+1$ and $(k+1)'$ to subdivide the edges $[k-1, k]$ and $[(k-1)', k']$, respectively. The resultant graph has a one-face embedding. Thus we may add the edge $[k+1, (k+1)']$ in such a way that the new graph R_{2k+2} has a 2-cell embedding with two faces, and is thus upper embeddable.

If $k+1$ is odd, R_{2k} has a 2-cell embedding with two faces. Since the Betti number of R_{2k+2} is even, we need its maximal embedding to have one face. Before inserting the vertices $k+1$ and $(k+1)'$, we establish the following claim.

Claim. *In the maximal embedding for R_{2k}, there is a vertex "b" such that the directed edges $(b, b+1)$ and $(b+1, b)$ are in different faces.*

Proof of claim. Suppose the opposite. Since the cycle 1, 2, 3, ..., k, 1 cannot be the boundary of a face in this embedding, there is a vertex z such that $(z, z+1)$ and $(z-1, z)$ are in different faces F and F', respectively. By assumption, $(z+1, z)$ is in F and $(z, z-1)$ is in F'. Because $(z-1, z)$ and $(z, z+1)$ are in different faces, and since z only has degree three, the vertex permutation P_z must give $P_z(z-1) = z'$. Likewise,

$P_z(z+1) = z'$ since $(z+1, z)$ and $(z, z-1)$ are in different faces. Since P_z is a one-one function, this is a contradiction.

In the maximal embedding of R_{2k}, use the vertices $(k+1)$ and $(k+1)'$ to subdivide the edges $[b, b+1]$ and $[b', (b+1)']$, respectively. Thus, by the claim, $(k+1)$ is in both faces of the two-face embeddings of this new graph. Hence we may add the edge $[k+1, (k+1)']$ so that the resultant graph has a one-face embedding. A relabeling of this graph gives R_{2k+2} as previously labeled. By induction, R_{2n} is upper embeddable.

In [5], the authors show that the maximum genus of a graph is at least as large as the sum of the maximum genera of its blocks. The proof used in that paper implies the following result.

Lemma. *Let G and H be disjoint connected graphs and let G' be the graph formed by identifying one vertex of G with one vertex of H. Then $\gamma_M(G') \geq \gamma_M(G) + \gamma_M(H)$.*

Theorem 2. *Let G be a connected graph of genus n and let m be any positive integer. Then there is a connected graph G', containing G as a subgraph, such that $\gamma(G') = n$, and $\gamma_M(G') > mn$.*

Proof. If $\gamma_M(G) > mn$, we are finished. Assume $\gamma_M(G) \leq mn$ and let $k = mn - \gamma_M(G)$. By Theorem 1, there is an integer j such that $\gamma_M(W_j) > k$. Let G' be the graph formed by identifying any vertex of G with any non-center vertex of W_j. Then $\gamma(G') = n$, since W_j is planar and may thus be placed in a 2-cell face of the genus embedding for G. Using the lemma, we have

$$\gamma_M(G') \geq \gamma_M(G) + \gamma_M(W_j) > mn\,.$$

Corollary. *Given arbitrary positive integers m and n, there is a connected graph of genus n and maximum genus larger than mn.*

Proof. In [1], the authors have shown that the genus of a graph is the sum of the genera of its blocks. Hence n copies of any toroidal graph may be combined to produce a graph of genus n. The corollary now follows from Theorem 2.

References

[1] J. Battle, F. Harary, Y. Kodama and J.W.T. Youngs, Additivity of the genus of a graph, Bull. Am. Math. Soc. 68 (1962) 565–568.

[2] J. Edmonds, A combinatorial representation for polyhedral surfaces, Notices Am. Math. Soc. 7 (1960) 646.

[3] F. Harary, Graph theory (Addison–Wesley, Reading, Mass., 1969).

[4] E. Nordhaus, R. Ringeisen, B. Stewart and A. White, A Kuratowski-type theorem for the maximum genus of a graph, J. Combin. Theory 12 (1972) 260–267.

[5] E. Nordhaus, B. Stewart and A. White, On the maximum genus of a graph, J. Combin. Theory 11 (1971) 258–267.

[6] R. Ringeisen, Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell embeddings, J. Combin. Theory 12 (1972) 101–104.

[7] J.W.T. Youngs, Minimal imbeddings and the genus of a graph, J. Math. Mech. 12 (1963) 303–315.